

Rigorous proof for the non-local correlation function in the antiferromagnetic seamed transverse Ising ring

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An unusual correlation function is conjectured by M. Campostrini *et al.* (Phys. Rev. E **91**, 042123 (2015)) for the ground state of a transverse Ising chain with geometrical frustration in one of the translationally invariant cases. Later, we demonstrated the correlation function and showed its non-local nature in the thermodynamic limit based on the rigorous evaluation of a Toeplitz determinant (J. Stat. Mech. 113102 (2016)). In this paper, we prove rigorously that all the states that forming the lowest gapless spectrum (including the ground state) in the kink phase exhibit the same asymptotic correlation function. So, in a point of view of canonical ensemble, the thermal correlation function is inert to temperature within the energy range of the lowest gapless spectrum.

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I. INTRODUCTION

Quantum antiferromagnetic spin chain is one of the subjects in quantum magnetism endowed with rich and interesting physics [1, 2]. In recent years, spin chains with deliberate boundary conditions that induce geometrical frustration has been getting more and more attention [3–6]. The simplest and fundamental model providing many essential physics of interest may be the transverse field Ising chain [7–11]

$$H_s = \sum_{j=1}^N J_j \sigma_j^x \sigma_{j+1}^x - \sum_{j=1}^N h_j \sigma_j^z, \quad (1)$$

with Pauli matrices σ_j^α ($\alpha = x, z$), exchange coupling J_j and external field h_j . When special boundary conditions are imposed, unusual properties may emerge due to geometrical frustration of spin arrangement [12–14]. In treating the ferromagnetic transverse Ising chain seamed by one antiferromagnetic bond, Campostrini *et al.* [13] conjectured an usual correlation function for the ground state based on numerical calculations. Later, in the context of translational invariance and the "a-cycle problem" [7], we analysed the antiferromagnetic seamed system with $J_j = J > 0$, $h_j = h$ and $N \in \text{Odd}$ [15], which is equivalent to one of the translational invariant case of the model considered by Campostrini *et al.* [13]. And based on rigorous solutions, we proved the conjectured correlation function, which exhibits a non-local nature in the thermodynamic limit [15].

Nonetheless, as disclosed by the rigorous solutions, there are $2N - 1$ similar low-lying quantum states above the ground state [15], which form a gapless spectrum in the kink phase ($h < J$) in the thermodynamic limit [13].

In this paper, we prove rigorously that the correlation functions of all the $2N$ quantum states (including the ground state) in the gapless spectrum have the same unusual asymptotic behaviour in the thermodynamic limit.

The organization of the paper is as follows. In Sec. II, the low-lying $2N$ quantum states of the a-cycle problem with ring frustration are reviewed. In Sec. III, we show how the correlation functions are linked to a special type of Toeplitz determinant. In Sec. IV, we give the rigorous proof of a generalized theorem for evaluating of the Toeplitz determinant. At last, the unusual correlation function is worked out in Sec. V.

II. THE LOW-LYING $2N$ QUANTUM STATES

Here we briefly review the low-lying $2N$ quantum states of the frustrated ring that we obtained in our previous work [15].

First, the spin ring is mapped to a model of spinless fermions by the Jordan-Wigner transformation [16]

$$\sigma_j^\pm = (\sigma_j^x \pm i \sigma_j^y) / 2 = c_j^\dagger \exp(i \pi \sum_{l < j} c_l^\dagger c_l). \quad (2)$$

Then Eq. (1) takes the form

$$H_f = Nh - 2h \sum_{j=1}^N c_j^\dagger c_j + J \sum_{j=1}^{N-1} (c_j^\dagger - c_j)(c_{j+1}^\dagger - c_{j+1}) - J \exp(i \pi M) (c_N^\dagger - c_N)(c_1^\dagger + c_1), \quad (3)$$

where $M = \sum_{l=1}^N c_l^\dagger c_l$. It has been shown that the last term in Eq. (3) can not be discarded for the frustrated ring, i.e. $J > 0$ and $N \in \text{odd}$. And the system should be solved faithfully in the framework of "a-cycle problem" [15, 17]. In the fermion representation, the Hilbert space of the Hamiltonian, Eq. (3), is enlarged twice as the one of the original spin Hamiltonian, Eq. (1). The quantum

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states of Eq. (3) are grouped into two channels, the odd and even channels labelled by $M \in \text{odd}$ and $M \in \text{even}$ respectively. To restore the full degrees of freedom (DOF) of the original spin Hamiltonian, we should project out half redundant DOF in each channel [18].

Second, the Hamiltonian Eq. (3) is easily solved with the help of parity constraint, Fourier transformation $c_q = \frac{1}{\sqrt{N}} \sum_{j=1}^N c_j \exp(i q j)$ and Bogoliubov transformation ($q \neq 0$ and π)

$$\eta_q = u_q c_q - i v_q c_{-q}^\dagger \quad (4)$$

with

$$\begin{aligned} u_q^2 &= \frac{1}{2} \left(1 + \frac{\epsilon(q)}{\omega(q)} \right), v_q^2 = \frac{1}{2} \left(1 - \frac{\epsilon(q)}{\omega(q)} \right), \\ 2u_q v_q &= \frac{\Delta(q)}{\omega(q)}, \epsilon(q) = J \cos q - h, \Delta(q) = J \sin q, \\ \omega(q) &= \sqrt{h^2 + J^2 - 2hJ \cos q}. \end{aligned} \quad (5)$$

The momentum values $q = 0$ and π play an important role in controlling the parity of valid spin states. In momentum space, the spin Hamiltonian can be expressed as [19, 20]

$$H_s = PH_f^{(e)}P \oplus PH_f^{(o)}P, \quad (6)$$

where $PH_f^{(e)}P$ means only even number fermionic occupation states are valid states for the original spin model due to the even parity constraint, while $PH_f^{(o)}P$ implies that only odd number fermionic occupation states are valid states, and P means projection of the redundant DOF [18]. $H_f^{(e)}$ and $H_f^{(o)}$ are given by

$$H_f^{(e)} = \epsilon(\pi) (2c_\pi^\dagger c_\pi - 1) + \sum_{q \in q^{(e)}, q \neq \pi} \omega(q) (2\eta_q^\dagger \eta_q - 1), \quad (7)$$

$$H_f^{(o)} = \epsilon(0) (2c_0^\dagger c_0 - 1) + \sum_{q \in q^{(o)}, q \neq 0} \omega(q) (2\eta_q^\dagger \eta_q - 1), \quad (8)$$

with

$$q^{(e)} = \left\{ -\frac{N-2}{N}\pi, \dots, -\frac{1}{N}\pi, \frac{1}{N}\pi, \dots, \frac{N-2}{N}\pi, \pi \right\}, \quad (9)$$

$$q^{(o)} = \left\{ -\frac{N-1}{N}\pi, \dots, -\frac{2}{N}\pi, 0, \frac{2}{N}\pi, \dots, \frac{N-1}{N}\pi \right\}. \quad (10)$$

In the kink phase ($h < J$), it is easy to write down the $2N$ low-lying states of the gapless spectrum of width $4h$. The ground state comes from the odd channel and reads

$$|E_0^{(o)}\rangle = c_0^\dagger |\phi^{(o)}\rangle. \quad (11)$$

The upper-most state comes from the even channel and reads

$$|E_\pi^{(e)}\rangle = |\phi^{(e)}\rangle. \quad (12)$$

The rest $2(N-1)$ states interweave between $|E_0\rangle$ and $|E_\pi\rangle$. Half of them are of odd parity

$$|E_k^{(o)}\rangle = \eta_k^\dagger |\phi^{(o)}\rangle, \left\{ k \in q^{(o)} | k \neq 0 \right\}, \quad (13)$$

and half of them are of even parity

$$|E_k^{(e)}\rangle = \eta_k^\dagger c_\pi^\dagger |\phi^{(e)}\rangle, \left\{ k \in q^{(e)} | k \neq \pi \right\}. \quad (14)$$

In the above, the BCS-type states

$$|\phi^{(o)}\rangle = \prod_{q \in q^{(o)}, 0 < q < \pi} (u_q + i v_q c_q^\dagger c_{-q}^\dagger) |0\rangle, \quad (15)$$

$$|\phi^{(e)}\rangle = \prod_{q \in q^{(e)}, 0 < q < \pi} (u_q + i v_q c_q^\dagger c_{-q}^\dagger) |0\rangle, \quad (16)$$

are vacuums corresponding to $H_f^{(e)}$ and $H_f^{(o)}$ respectively, and $|0\rangle = |\downarrow\downarrow\downarrow \dots \downarrow\rangle$.

There is a rough but nice picture for the low-lying states in a perturbative treatment, which gives the following superposed states [15],

$$|A_p\rangle = \frac{1}{\sqrt{2N}} \sum_j e^{-ipj} (|K(j), \leftarrow\rangle + |K(j), \rightarrow\rangle), \quad (17)$$

$$|B_p\rangle = \frac{1}{\sqrt{2N}} \sum_j e^{-ipj} (|K(j), \leftarrow\rangle - |K(j), \rightarrow\rangle), \quad (18)$$

$$p = \left\{ -\frac{N-1}{N}\pi, \dots, -\frac{2}{N}\pi, 0, \frac{2}{N}\pi, \dots, \frac{N-1}{N}\pi \right\}. \quad (19)$$

They are translational invariant states composed of the classical kink states,

$$\begin{aligned} |K(j), \rightarrow\rangle &= |\dots, \leftarrow_{j-1}, \rightarrow_j, \rightarrow_{j+1}, \leftarrow_{j+2}, \dots\rangle, \\ |K(j), \leftarrow\rangle &= |\dots, \rightarrow_{j-1}, \leftarrow_j, \leftarrow_{j+1}, \rightarrow_{j+2}, \dots\rangle. \end{aligned} \quad (20)$$

One can find the correspondence between the exact states and the approximate states ($h \ll J$)

$$|E_0^{(o)}\rangle \approx |A_0\rangle, \quad |E_\pi^{(e)}\rangle \approx |B_0\rangle, \quad (21)$$

$$|E_q^{(o)}\rangle \approx |A_q\rangle, \quad |E_q^{(e)}\rangle \approx |B_{\pi-q}\rangle. \quad (22)$$

III. LONGITUDINAL CORRELATION FUNCTIONS

The two point longitudinal spin-spin correlation function for arbitrary state $|\psi\rangle$ is defined as

$$C_{r,N}^{xx}(|\psi\rangle) = \langle \psi | \sigma_j^x \sigma_{j+r}^x | \psi \rangle. \quad (23)$$

Due to the translational invariance, the correlation function depends on the separation r between two spins rather than the lattice position j . And because of the periodic boundary condition, we have a cyclic relation,

$C_{r,N}^{xx}(|\psi\rangle) = C_{N-r,N}^{xx}(|\psi\rangle)$, for a translationally invariant state $|\psi\rangle$.

By the approximate states, Eq. (17)-(18), we can easily get

$$C_{r,N}^{xx}(|A_p\rangle) = C_{r,N}^{xx}(|B_p\rangle) = (-1)^r \left(1 - \frac{2r}{N}\right). \quad (24)$$

When approaching the thermodynamic limit $N \rightarrow \infty$, one can set $r/N \rightarrow 0$ to see a local correlation or set a finite r/N to see a non-local correlation. Since the approximate states are valid for $h \ll J$, so does Eq. (24).

A rigorous result for the exact ground state has been proved in our previous work, which reads ($N \gg 1$)

$$C_{r,N}^{xx}(|E_0^{(o)}\rangle) = (-1)^r \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left(1 - \frac{2r}{N}\right). \quad (25)$$

The same result was conjectured by Campostrini *et al.* in a context of the transverse Ising ring with one-bond defect [13]. Now we demonstrate that the rest $2N - 1$ low-lying states exhibit the same behavior.

First, we show that all of the correlation functions of the $2N$ low-lying states can be casted into a general Toeplitz determinant. Comparing with the case of ground state, the general Toeplitz determinant is much more complicated and depends on the wave number k . We should also be careful to the parity channels.

The correlation function of the ground state $|E_0^{(o)}\rangle = c_0^\dagger |\phi^{(o)}\rangle$ is given by a determinant as

$$\begin{aligned} C_{r,N}^{xx}(|E_0^{(o)}\rangle) &= \langle \phi^{(o)} | c_0 B_j A_{j+1} \dots B_{j+r-1} A_{j+r} c_0^\dagger | \phi^{(o)} \rangle \\ &= \begin{vmatrix} D_0 + \frac{2}{N} & D_{-1} + \frac{2}{N} & \dots & D_{1-r} + \frac{2}{N} \\ D_1 + \frac{2}{N} & D_0 + \frac{2}{N} & \dots & D_{2-r} + \frac{2}{N} \\ \dots & \dots & \dots & \dots \\ D_{r-1} + \frac{2}{N} & D_{r-2} + \frac{2}{N} & \dots & D_0 + \frac{2}{N} \end{vmatrix}, \end{aligned} \quad (26)$$

where, following Lieb [7], we have introduced $A_j = c_j^\dagger + c_j$ and $B_j = c_j^\dagger - c_j$. And we have making use of the Wick's theorem and the contractions in respect of $|\phi^{(o)}\rangle$: $\langle A_l A_m \rangle = \delta_{lm}$, $\langle c_0 c_0^\dagger \rangle = 1$, $\langle B_l B_m \rangle = -\delta_{lm}$, $\langle A_j c_0^\dagger \rangle = -\langle B_j c_0^\dagger \rangle = \frac{1}{\sqrt{N}}$, $\langle B_l A_m \rangle = D_{l-m+1}$, with $D_n = \frac{1}{N} \sum_{q \in q^{(o)}} D(e^{iq}) \exp(-iqn)$ and

$$D(e^{iq}) = -\frac{J - h e^{-iq}}{\sqrt{(J - h e^{-iq})(J - h e^{iq})}}. \quad (27)$$

For the $N - 1$ states from odd parity $|E_k^{(o)}\rangle = \eta_k^\dagger |\phi^{(o)}\rangle$, $\{k \in q^{(o)} | k \neq 0\}$, we can arrive at

$$\begin{aligned} C_{r,N}^{xx}(|E_k^{(o)}\rangle) &= \langle \phi^{(o)} | \eta_k B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_k^\dagger | \phi^{(o)} \rangle \\ &= \frac{1}{2} \left[\Gamma^{(o)}(r, N, \alpha_k, e^{ik}) + \Gamma^{(o)}(r, N, \alpha_{-k}, e^{-ik}) \right], \end{aligned} \quad (28)$$

where

$$\Gamma^{(o)}(r, N, \alpha_k, e^{ik}) = \begin{vmatrix} D_0 + \frac{2\alpha_k}{N} & D_{-1} + \frac{2\alpha_k}{N} e^{-ik} & \dots & D_{1-r} + \frac{2\alpha_k}{N} e^{i(1-r)k} \\ D_1 + \frac{2\alpha_k}{N} e^{ik} & D_0 + \frac{2\alpha_k}{N} & \dots & D_{2-r} + \frac{2\alpha_k}{N} e^{i(2-r)k} \\ \dots & \dots & \dots & \dots \\ D_{r-1} + \frac{2\alpha_k}{N} e^{i(r-1)k} & D_{r-2} + \frac{2\alpha_k}{N} e^{i(r-2)k} & \dots & D_0 + \frac{2\alpha_k}{N} \end{vmatrix}, \quad (29)$$

and $\langle \eta_k B_l \rangle \langle A_m \eta_k^\dagger \rangle = \frac{\alpha_k}{N} \exp(i k (l - m + 1))$, $\alpha_k \equiv -D(e^{ik})$.

Next, we consider the $N - 1$ states from even parity $|E_k^{(e)}\rangle = \eta_k^\dagger c_\pi^\dagger |\phi^{(e)}\rangle$, $\{k \in q^{(e)} | k \neq \pi\}$. Similarly, Wick's theorem can be applied and we can arrive at

$$\begin{aligned} C_{r,N}^{xx}(|E_k^{(e)}\rangle) &= \langle \phi^{(e)} | c_\pi \eta_k B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_k^\dagger c_\pi^\dagger | \phi^{(e)} \rangle \\ &= \frac{1}{2} \left[\Gamma^{(e)}(r, N, \alpha_k, e^{ik}) + \Gamma^{(e)}(r, N, \alpha_{-k}, e^{-ik}) \right], \end{aligned} \quad (30)$$

where

$$\Gamma^{(e)}(r, N, \alpha_k, e^{ik}) = \begin{vmatrix} F_0 + \frac{2\alpha_k}{N} & F_{-1} + \frac{2\alpha_k}{N} e^{-ik} & \cdots & F_{1-r} + \frac{2\alpha_k}{N} e^{i(1-r)k} \\ F_1 + \frac{2\alpha_k}{N} e^{ik} & F_0 + \frac{2\alpha_k}{N} & \cdots & F_{2-r} + \frac{2\alpha_k}{N} e^{i(2-r)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{r-1} + \frac{2\alpha_k}{N} e^{i(r-1)k} & F_{r-2} + \frac{2\alpha_k}{N} e^{i(r-2)k} & \cdots & F_0 + \frac{2\alpha_k}{N} \end{vmatrix}, \quad (31)$$

$$F_n = \frac{1}{N} \sum_{q \in q^{(e)}} \exp(-iqn) \frac{-(J - h e^{iq})}{\sqrt{(J - h e^{-iq})(J - h e^{iq})}}. \quad (32)$$

Finally, for the upper-most level $|E_\pi^{(e)}\rangle = |\phi^{(e)}\rangle$, after the contractions with $|\phi^{(e)}\rangle$ we arrive at

$$\begin{aligned} C_{r,N}^{xx}(|E_\pi^{(e)}\rangle) &= \langle \phi^{(e)} | B_j A_{j+1} \cdots A_{j+r-1} B_{j+r-1} A_{j+r} | \phi^{(e)} \rangle \\ &= \begin{vmatrix} F_0 + \frac{2}{N} & F_{-1} + \frac{2}{N} e^{-i\pi} & \cdots & F_{1-r} + \frac{2}{N} e^{i\pi(1-r)} \\ F_1 + \frac{2}{N} e^{i\pi} & F_0 + \frac{2}{N} & \cdots & F_{2-r} + \frac{2}{N} e^{i\pi(2-r)} \\ \cdots & \cdots & \cdots & \cdots \\ F_{r-1} + \frac{2}{N} e^{i\pi(r-1)} & F_{r-2} + \frac{2}{N} e^{i\pi(r-2)} & \cdots & F_0 + \frac{2}{N} \end{vmatrix}. \end{aligned} \quad (33)$$

For large enough N , we have

$$D_n = F_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \frac{-(J - h e^{iq})}{\sqrt{(J - h e^{-iq})(J - h e^{iq})}}. \quad (34)$$

So the correlation functions for the $2N$ low-lying states, Eq. (26), Eq. (28), Eq. (30) and Eq. (33), can be written in a uniform formula in the thermodynamic limit,

$$C_{r,N}^{xx}(|E_k^{(o/e)}\rangle) = \frac{1}{2} [\Gamma(r, N, \alpha_k, e^{ik}) + \Gamma(r, N, \alpha_{-k}, e^{-ik})], \quad (35)$$

where

$$\Gamma(r, N, \alpha_k, e^{ik}) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_{-1} & \cdots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \cdots & \tilde{D}_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{D}_{r-1} & \tilde{D}_{r-2} & \cdots & \tilde{D}_0 \end{vmatrix}, \quad (36)$$

$$\tilde{D}_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \frac{-(J - h e^{iq})}{\sqrt{(J - h e^{-iq})(J - h e^{iq})}} + \frac{2\alpha_k}{N} e^{ikn}, \quad (37)$$

IV. GENERALIZED THEOREM

Theorem: Consider a Toeplitz determinant

$$\Theta(r, N, x, e^{ik}) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_{-1} & \cdots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \cdots & \tilde{D}_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{D}_{r-1} & \tilde{D}_{r-2} & \cdots & \tilde{D}_0 \end{vmatrix} \quad (38)$$

with

$$\tilde{D}_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} D(e^{iq}) e^{-iqn} + \frac{x}{N} e^{ikn}. \quad (39)$$

Due to the presence of the extra term $\frac{2\alpha_k}{N} e^{ikn}$ in each element of the new Toeplitz determinant, Eq. (36), the Szegő limit theorem cannot be applied directly [21, 22]. To evaluate the correlation functions analytically, we need to prove a generalized theorem for this special determinant.

If the generating function $D(e^{iq})$ and $\ln D(e^{iq})$ are continuous on the unit circle $|e^{iq}| = 1$, then the behavior for

large N of $\Theta(r, N, x, e^{ik})$ is given by ($1 \ll r < N$)

$$\Theta(r, N, x, e^{ik}) = \Delta_r \left(1 + \frac{xr}{ND(e^{-ik})} \right), \quad (40)$$

where

$$\Delta_r = \mu^r \exp\left(\sum_{n=1}^{\infty} nd_{-n}d_n\right), \quad (41)$$

$$\mu = \exp\left[\int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln D(e^{iq})\right], \quad (42)$$

$$d_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \ln D(e^{iq}), \quad (43)$$

if the sum in Eq. (41) converges.

Proof: Let $e^{iq} = \xi$, $D_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} D(\xi)\xi^{-n}$, then $\tilde{D}_n = D_n + \frac{x}{N} e^{ikn}$. First, we rewrite Eq. (38) as

$$\begin{aligned} \Theta(r, N, x, e^{ik}) = & \begin{vmatrix} D_0 & D_{-1} & \cdots & D_{-r+1} \\ D_1 & D_0 & \cdots & D_{-r+2} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & D_0 \end{vmatrix} + \begin{vmatrix} \frac{x}{N} & D_{-1} & \cdots & D_{1-r} \\ \frac{x}{N} e^{ik} & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x}{N} e^{i(r-1)k} & D_{r-2} & \cdots & D_0 \end{vmatrix} \\ & + \cdots + \begin{vmatrix} D_0 & \frac{x}{N} e^{-ik} & \cdots & D_{2-r} \\ D_1 & \frac{x}{N} & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & \frac{x}{N} e^{i(r-2)k} & \cdots & D_0 \end{vmatrix} + \begin{vmatrix} D_0 & D_{-1} & \cdots & \frac{x}{N} e^{i(1-r)k} \\ D_1 & D_0 & \cdots & \frac{x}{N} e^{i(2-r)k} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & \frac{x}{N} \end{vmatrix}. \end{aligned}$$

Then we compose a set of linear equations

$$\sum_{m=0}^{r-1} D_{n-m} x_m^{(r-1)} = \frac{x}{N} e^{ikn}, \quad 0 \leq n \leq r-1. \quad (44)$$

These equations have an unique solution for $x_n^{(r-1)}$ if there exists a non-zero determinant:

$$\Delta_r \equiv \begin{vmatrix} D_0 & D_{-1} & \cdots & D_{1-r} \\ D_1 & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & D_0 \end{vmatrix} \neq 0. \quad (45)$$

By Cramer's rule, we have the solution:

$$x_0^{(r-1)} = \frac{\begin{vmatrix} \frac{x}{N} & D_{-1} & \cdots & D_{1-r} \\ \frac{x}{N} e^{ik} & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x}{N} e^{i(r-1)k} & D_{r-2} & \cdots & D_0 \end{vmatrix}}{\Delta_r}, \quad (46)$$

$$x_1^{(r-1)} = \frac{\begin{vmatrix} D_0 & \frac{x}{N} e^{-ik} & \cdots & D_{2-r} \\ D_1 & \frac{x}{N} & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & \frac{x}{N} e^{i(r-2)k} & \cdots & D_0 \end{vmatrix}}{\Delta_r}, \quad (47)$$

\vdots

$$x_{r-1}^{(r-1)} = \frac{\begin{vmatrix} D_0 & D_{-1} & \cdots & \frac{x}{N} e^{i(1-r)k} \\ D_1 & D_0 & \cdots & \frac{x}{N} e^{i(2-r)k} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & \frac{x}{N} \end{vmatrix}}{\Delta_r}. \quad (48)$$

So we arrive at

$$\Theta(r, N, x, e^{ik}) = \Delta_r + \Delta_r \sum_{n=0}^{r-1} e^{-ikn} x_n^{(r-1)}. \quad (49)$$

For our problem, Δ_r can be evaluated directly by using Szegő's theorem, so we need to know how to calculate the second term in Eq. (49). Follow the standard Wiener-Hopf procedure [22–24], we consider a generalization of

Eq. (44)

$$\sum_{m=0}^{r-1} D_{n-m} x_m = y_n, \quad 0 \leq n \leq r-1 \quad (50)$$

and define

$$x_n = y_n = 0 \quad \text{for } n \leq -1 \quad \text{and } n \geq r \quad (51)$$

$$\begin{aligned} v_n &= \sum_{m=0}^{r-1} D_{n-m} x_m \quad \text{for } n \geq 1 \\ &= 0 \quad \text{for } n \leq 0 \end{aligned} \quad (52)$$

$$\begin{aligned} u_n &= \sum_{m=0}^{r-1} D_{r-1+n-m} x_m \quad \text{for } n \geq 1 \\ &= 0 \quad \text{for } n \leq 0 \end{aligned} \quad (53)$$

We further define

$$\begin{aligned} D(\xi) &= \sum_{n=-\infty}^{\infty} D_n \xi^n, \quad Y(\xi) = \sum_{n=0}^{r-1} y_n \xi^n, \\ V(\xi) &= \sum_{n=1}^{\infty} v_n \xi^n, \quad U(\xi) = \sum_{n=1}^{\infty} u_n \xi^n, \\ X(\xi) &= \sum_{n=0}^{r-1} x_n \xi^n. \end{aligned} \quad (54)$$

It then follows from Eq. (50) that we can get

$$D(\xi) X(\xi) = Y(\xi) + V(\xi^{-1}) + U(\xi) \xi^{r-1} \quad (55)$$

for $|\xi| = 1$. Because $D(\xi)$ and $\ln D(\xi)$ is continuous and periodic on the unit circle, $D(\xi)$ has a unique factorization, up to a multiplicative constant, in the form

$$D(\xi) = P^{-1}(\xi) Q^{-1}(\xi^{-1}), \quad (56)$$

for $|\xi| = 1$, such that $P(\xi)$ and $Q(\xi)$ are both analytic for $|\xi| < 1$ and continuous and nonzero for $|\xi| \leq 1$. we may now use the factorization of $D(\xi)$ in Eq. (55) to write

$$\begin{aligned} P^{-1}(\xi) X(\xi) &- [Q(\xi^{-1}) Y(\xi)]_+ \\ &- [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_+ \\ &= [Q(\xi^{-1}) Y(\xi)]_- + Q(\xi^{-1}) V(\xi^{-1}) \\ &+ [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_-, \end{aligned} \quad (57)$$

where the subscript $+$ ($-$) means that we should expand the quantity in the brackets into a Laurent series and keep only those terms where ξ is raised to a non-negative (negative) power. The left-hand side of Eq. (57) defines a function analytic for $|\xi| < 1$ and continuous on $|\xi| = 1$ and the right-hand side defines a function which is analytic for $|\xi| > 1$ and is continuous for $|\xi| = 1$. Taken

together they define a function $E(\xi)$ analytic for all ξ except possibly for $|\xi| = 1$ and continuous everywhere. But these properties are sufficient to prove that $E(\xi)$ is an entire function which vanished at $|\xi| = \infty$ and thus, by Liouville's theorem, must be zero everywhere [22, 23]. Therefore both the right-hand side and the left-hand side of Eq. (57) vanish separately and thus we have

$$\begin{aligned} X(\xi) &= P(\xi) [Q(\xi^{-1}) Y(\xi)]_+ \\ &+ P(\xi) [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_+. \end{aligned} \quad (58)$$

Furthermore, $U(\xi)$ can be neglected for large r

$$X(\xi) \approx P(\xi) [Q(\xi^{-1}) Y(\xi)]_+. \quad (59)$$

Consider the term $[Q(\xi^{-1}) Y(\xi)]_+$, because $Q(\xi)$ is a $+$ function, so we can expand it as a Laurent series and keep only those term where ξ is raised to a non-negative power,

$$\begin{aligned} Q(\xi) &= \sum_{n=0}^{\infty} a_n \xi^n \\ &= (a_0 + a_1 \xi^1 + a_2 \xi^2 + \dots + a_{r-1} \xi^{r-1}) + O(\xi^r), \end{aligned} \quad (60)$$

and then

$$Q(\xi^{-1}) = a_0 + a_1 \xi^{-1} + a_2 \xi^{-2} + \dots + a_{r-1} \xi^{1-r}, \quad (61)$$

where we have neglected the term $O(\xi^r)$ for large r for clarity, from Eq. (44) and Eq. (54), we have

$$\begin{aligned} Y(\xi) &= \sum_{n=0}^{r-1} y_n \xi^n \\ &= \frac{x}{N} (1 + e^{ik} \xi^1 + e^{2ik} \xi^2 + \dots + e^{i(r-1)k} \xi^{r-1}), \end{aligned} \quad (62)$$

thus

$$\begin{aligned} [Q(\xi^{-1}) Y(\xi)]_+ &= \frac{x}{N} [(a_0 + a_1 e^{ik} + a_2 e^{2ik} + \dots + a_{r-1} e^{i(r-1)k}) \\ &+ (a_0 e^{ik} + a_1 e^{2ik} + \dots + a_{r-2} e^{i(r-1)k}) \xi^1 \\ &+ \dots + (a_0 e^{i(r-1)k}) \xi^{r-1}]. \end{aligned} \quad (63)$$

From Eq. (49), Eq. (54) and Eq. (59), we have

$$\begin{aligned} \sum_{n=0}^{r-1} e^{-ikn} x_n^{(r-1)} &= X(e^{-ik}) \\ &= P(e^{-ik}) [Q(e^{ik}) Y(e^{-ik})]_+, \end{aligned} \quad (64)$$

and

$$\begin{aligned}
& [Q(e^{ik}) Y(e^{-ik})]_+ \\
&= \frac{x}{N} \left[ra_0 + ra_1 e^{ik} + ra_2 e^{2ik} + \dots + ra_{r-1} e^{i(r-1)k} \right] \\
&\quad - \frac{x}{N} \left[a_1 e^{ik} + 2a_2 e^{2ik} + \dots + (r-1)a_{r-1} e^{i(r-1)k} \right] \\
&= \frac{x}{N} \left[rQ(e^{ik}) - e^{ik} \frac{dQ(\xi)}{d\xi} \Big|_{\xi=e^{ik}} \right]. \quad (65)
\end{aligned}$$

So when $r \gg 1$, we can ignore the second term in Eq. (65). Together with Eq. (56), we get

$$X(e^{-ik}) = \frac{xr}{N} P(e^{-ik}) Q(e^{ik}) = \frac{xr}{ND(e^{-ik})}. \quad (66)$$

At last, by Szegő's theorem, we get

$$\Delta_r = \mu^r \exp\left(\sum_{n=1}^{\infty} nd_{-n}d_n\right), \quad (67)$$

where

$$\begin{aligned}
\mu &= \exp \left[\int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln D(e^{iq}) \right], \\
d_n &= \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \ln D(e^{iq}). \quad (68)
\end{aligned}$$

From Eq. (49), we have

$$\Theta(r, N, x, e^{ik}) = \Delta_r \left(1 + \frac{xr}{ND(e^{-ik})} \right). \quad (69)$$

Q.E.D.

V. EVALUATION OF THE CORRELATION FUNCTIONS

Now we can evaluate $\Gamma(r, N, \alpha_k, e^{ik})$ in Eq. (36) by applying the *Theorem* directly to get

$$\begin{aligned}
\Gamma(r, N, \alpha_k, e^{ik}) &= \Gamma(r, N, \alpha_{-k}, e^{-ik}) \\
&= (-1)^r \left(1 - \frac{h^2}{J^2} \right)^{1/4} \left(1 - \frac{2r}{N} \right), \quad (70)
\end{aligned}$$

for large r and N . So the asymptotic behavior of the correlation functions is given by $(k \in q^{(o)} \cup q^{(e)})$

$$C_{r,N}^{xx}(|E_k^{(o/e)}\rangle) = (-1)^r \left(1 - \frac{h^2}{J^2} \right)^{1/4} \left(1 - \frac{2r}{N} \right), \quad (71)$$

which is independent of k . Thus, the correlation functions of the $2N$ low-lying states possess the same asymptotic behavior. If assuming the canonical ensemble for the low-lying energy levels, one would agree that the $2N$ levels dominate the system's properties at low temperatures ($T \ll 4h/k_B$, k_B is the Boltzmann constant) and arrive at a conclusion that the thermal correlation function is inert to temperature meanwhile.

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